DENSE SINGLE-VALUEDNESS OF MONOTONE OPERATORS

BY

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ABSTRACT

It is shown that the set of points for which a monotone mapping $T: X \to X^*$ from a separable Banach space into its dual is not single-valued has no interior; if $\dim X < \infty$ and $\operatorname{int} D(T) \neq \phi$ then the set has Lebesgue measure zero. Moreover, for accretive mappings $T: X \to X$ from a separable Banach space into itself, the dimension of the set of points whose images contain balls of codimension not larger than k does not exceed k. Applications to convexity are given.

A well-known theorem due to S. Mazur [4], [2, V. 9.8] says that the boundary of a closed convex body in a separable Banach space is smooth (i.e. has a tangent plane) at a dense set of points. In finite dimensional spaces [1], and to a certain extent in Hilbert space [6], more can be said in the sense that if the boundary points are classified according to smoothness then their relative abundance increases with smoothness. Equivalent statements can be given in terms of differentiability properties of convex functions. An inspection of the proofs reveals that underlying these geometrical facts there are basic theorems concerning the degree of multivaluedness of monotone operators. To present these theorems is the purpose of this article; the proofs are modelled after those for the corresponding properties of convex sets [2, V.9.8], [6, Th. 2.1].

Let us briefly go over some standard definitions. A set M in the Cartesian product $X \times X^*$ of a real Banach space with its dual is said to be monotone if

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geqq 0, \ \forall (x_1, x_1^*), \ (x_2, x_2^*) \in M$$

($\langle x^*, x \rangle$ denotes the value of the linear functional x^* at x). A maximal monotone

Received August 16, 1972

set is one not properly contained in another monotone set. A set valued mapping $T: X \to 2^{X^*}$ is called a monotone operator if its graph

$$\{(x^*, x) \mid x^* \in Tx\}$$

is a monotone set in $X \times X^*$; the operator is said to be maximal monotone if its graph is maximal monotone. For a given $T: X \to 2^{X^*}$ the operator T^{-1} is defined as the mapping from X^* into X having as its graph the set $\{(x, x^*) \mid (x^*, x) \in \text{graph of } T\}$. It is clear that T and T^{-1} are simultaneously monotone or maximal monotone. In the sequel we shall view set-valued mappings as multivalued mappings and use the notation $T: X \to X^*$ rather than $T: X \to 2^{X^*}$.

THEOREM 1. *The set of points where a monotone operator $T: X \to X^*$ from a separable Banach space into its dual is not single valued has an empty interior. If the domain of T has a nonempty interior the set is an F_σ-set; if in addition X is a finite dimensional the set has a Lebesgue measure zero.*

PROOF. It is clear that the theorem holds for T whenever it holds for any of its extensions. Hence, since any monotone mapping admits a maximal extension, it may be assumed without loss of generality that T is maximal monotone. Let $D(T)$ denote its domain of definition. We may further assume that $\operatorname{int} D(T) \neq \phi$, for otherwise there is nothing to prove. Under such conditions a theorem of R. T. Rockafellar [5, Th. 1] says that $\operatorname{int} D(T)$ is an open convex set whose closure contains $D(T)$, and at any of whose points T is locally bounded. In particular, the image Tx of any x in $\operatorname{int} D(T)$ is a bounded set in X^*; by the maximality it is also closed and convex.

The theorem will be proved as soon as it is shown that

$$Z = \{x \in \operatorname{int} D(T) \mid Tx \text{ not a singleton}\}$$

contains no nonempty open set. Let us consider the real valued function

$$k(x, u) = \sup_{x^* \in Tx} \langle x^*, u \rangle, \quad x \in D(T), \; u \in X.$$

For fixed x, $k(x, u)$ is a lower semicontinuous convex function of u: the support function of the convex set Tx. If $x \in \operatorname{int} D(T)$, it is everywhere finite. On the other hand, for fixed u, it is an upper semicontinuous function of x on $\operatorname{int} D(T)$. To see this, pick, for any $x \in \operatorname{int} D(T)$, a sequence x_n converging to x, and in each Tx_n an x_n^* in such a fashion that $\langle x_n^*, u \rangle \to \limsup_{y \to x} k(y, u)$. By the local boundedness of T at x, $\{x_n^*\}_1^\infty$ is a bounded set, and the conditional weak-star compactness of bounded sets in X^* and the separability of X guarantee the existence of a subsequence $x^*_{n_k}$ converging weakly to a point x^* in X^*. By the maximality of T, $x^* \in Tx$. Then,

$$k(x,u) \geqq \langle x^*, u\rangle = \lim_{k\to\infty} \langle x^*_{n_k}, u\rangle = \lim_{n\to\infty} \langle x^*_n, u\rangle = \lim_{y\to x} \sup k(y,u),$$

and the upper semicontinuity of $k(x,u)$ with regard to x is established.

The important fact about $k(x,u)$ is that along any line parallel to u it is a monotone function of x. Let us look at this more closely. If $x \in \operatorname{int} D(T)$ the line $\{x + tu\}_{-\infty < t < +\infty}$ intersects int $D(T)$ in an open segment. Let s and t be two real numbers such that $s < t$ and $x + su$, $x + tu \in \operatorname{int} D(T)$. Then if $x^*_s \in T(x + su)$, $x^*_t \in T(x + tu)$,

$$\langle x^*_t, u\rangle - \langle x^*_s, u\rangle = (t-s)^{-1}\langle x^*_t - x^*_s, \quad (x+tu) - (x+su)\rangle \geqq 0,$$

by the monotonicity of T. Hence

$$k(x+tu,u) = \sup_{x^*_t \in T(x+tu)} \langle x^*_t, u\rangle \geqq \inf_{x^*_t \in T(x+tu)} \langle x^*_t, u\rangle = - \sup_{x^*_t \in T(x+tu)} \langle x^*_t, -u\rangle$$

$$= -k(x+tu, -u) \geqq \sup_{x^*_s \in T(x+su)} \langle x^*_s, u\rangle = k(x+su,u),$$

which makes it plain that $k(x + tu, u)$ is a nondecreasing function of t, and that

$$0 \leqq k(x+tu,u) + k(x+tu,-u) \leqq k(x+tu,u) - k(x+su,u).$$

Letting $s \uparrow t$ it follows that

(1) $$0 \leqq k(x+tu,u) + k(x+tu,-u) \leqq k(x+tu,u) - \lim_{s\uparrow t} k(x+su,u).$$

The quantity

$$k(x,u) + k(x,-u) = \sup_{x^* \in Tx} \langle x^*, u\rangle - \inf_{x^* \in Tx} \langle x^*, u\rangle$$

is the supremum of the lengths of the projections on u of all the differences of points in Tx. Thus, if $\{u_n\}_1^\infty \subset X$ is a sequence such that $\langle x^*, u_n\rangle = 0$ for all n implies $x^* = 0$, then Tx is not a singleton iff for at least one n $\langle Tx, u_n\rangle$ is not a a singleton, that is, iff $k(x,u_n) + k(x,-u_n) > 0$. Letting

$$Z_n = \{x \in \operatorname{int} D(T) \mid k(x,u_n) + k(x,-u_n) > 0\},$$

we have

(2) $$Z = \bigcup_{n=1}^{\infty} Z_n.$$

Since by (1) the points of Z_n on the line $\{x + tu_n\}_{-\infty < t < +\infty}$ are associated with jumps of the nondecreasing function of t, $k(x + tu_n, u_n), Z_n$ intersects any line parallel to u_n in at most a countable number of points. In particular, Z_n has an empty interior for every n. In the finite dimensional case, Fubini's theorem allows one to conclude from this that the Z_n's are all sets of Lebesgue measure zero, and hence that Z itself is of measure zero.

Clearly,

$$Z_n = \bigcup_{m=1}^{\infty} Z_{n,m} \tag{3}$$

where

$$Z_{n,m} = \{x \in \operatorname{int} D(T) \mid k(x, u_n) + k(x, -u_n) \geqq m^{-1}\}.$$

The upper semicontinuity of $k(x,u) + k(x, -u)$ with regard to the first argument indicates that all the $Z_{n,m}$'s are closed sets. Substituting (3) in (2),

$$Z = \bigcup_{n,m=1}^{\infty} Z_{n,m}.$$

Now, if Z had a nonempty interior then it would contain a closed ball B of non-vanishing radius, and for this ball

$$B = \bigcup_{n,m=1}^{\infty} (Z_{n,m} \cap B).$$

But, since B is of the second Baire category and the $(Z_{n,m} \cap B)$'s are closed, one of these sets, and hence a Z_n, would have to have a nonempty interior, which is impossible. Thus Z has no interior, and is an F_σ-set. Q.E.D.

REMARK. As the countable union of F_σ-sets is again of the same type it follows that if $\{T_n^\infty\}_1$ is a sequence of monotone operators simultaneously defined in an open set O, then the set of points where all the T_n's are simultaneously single-valued is dense in O. Another point to be observed is that the restriction of T to the set of points where T is single-valued is demicontinuous, and in consequence that T is demicontinuous in a dense set, if $\operatorname{int} \mathscr{D}(T)$ is not empty.

To a monotone operator $T: X \to X^*$, a closed linear subspace L, and point x_0 in X may be associated the mapping $T_L: L \to X^*/L^\perp \simeq L^*$ ($L^\perp$ is the annihilator of L) defined by $T_L x = \hat{T}(x + x_0)$, where $\hat{x}^*$ denotes the coset in $X^*/L^\perp$ containing x^*. This is a monotone operator and Theorem 1 applied to it yields:

COROLLARY 1. *Let $T: X \to X^*$ be a monotone operator and M a separable affine manifold in X. Then the set of points in M where Tx is not orthogonal to M has no interior in M; if M is finite dimensional the set has Lebesgue measure zero.*

In the corollary above, the orthogonality is to be understood as meaning that the difference of any two points in Tx annihilates the difference of any two points in M. If M is identified with the manifold spanned by Tx one obtains the following interesting consequence:

COROLLARY 2. *If $T: X \to X^*$ is a monotone operator and both X and X^* are separable then, for given x in X and any y in Tx with the exception of points in a set with empty interior, Tx and $T^{-1}y$ are orthogonal convex sets.*

When applied to the subgradient mapping $x \to \partial p_K(x)$ of the Minkowski functional of a closed convex set K Theorem 1 leads at once to the theorem of Mazur mentioned at the beginning of this article. The mapping is monotone and, by the Hahn-Banach Theorem, everywhere defined whenever the origin is an interior point [2, V.9].

A sort of dual of Mazur's result concerning the faces of a convex set can be described as follows: If K is a closed convex set in a Banach space X and u^* is a vector in the dual space X^*, we call the closed convex set

$$F_K u^* = \{x \mid x \in K, \langle u^*, x \rangle = \sup_{y \in K} \langle u^*, y \rangle\}$$

the face of K perpendicular to u^*, and we say that u^* is a (outer) normal to K at points of this face. Naturally the face may be empty; when it reduces to a point it is called an *exposed point.* The mapping $F_K: X^* \to X$ assigning to each $u^* \in X^*$ the corresponding face $F_K u^*$—called the support mapping of K—is a monotone operator defined over the set of normals to K. If X^* is separable, F_K satisfies the hypotheses of Theorem 1 and the following result may be derived:

COROLLARY 3. *The set of normals to a closed convex set K in a Banach space X with a separable dual X^* at more than one point has an empty interior; in finite dimensions, the set has measure zero. If X is reflexive and K bounded, the normals at exposed points are dense in X^*.*

Corollaries 1 and 2 also have known geometrical meanings. It is interesting to look at Theorem 1 in the context of the so called Fredholm Alternative. This proposition, valid for a certain type of linear operators, says, among other things, that if the equation

$$(4) \qquad Sx = y$$

is solvable for all y's in space, it is uniquely solvable, and conversely. Naturally, since we are dealing with nonlinear operators, the space here can be replaced by any open set. Now, if S is identified with the inverse of a monotone mapping, Theorem 1 can be read as saying that if (4) is solvable for all y's in an open set then it is uniquely solvable for a dense set therein. The resemblance is more striking still in finite dimensions where solvability in an open set implies uniqueness almost everywhere. A further analysis shows that Corollaries 1 and 2 have bearing on other aspects of the Fredholm alternative.

THEOREM 2. *Let $S: X \to Y$ be a Lipschitz mapping from a separable Banach space X into a Banach space Y. Then the set $W^{(k)} = \{y \in Y \mid S^{-1}y$ contains a relatively open ball of codimension less than or equal to $k\}$*† *is contained in the union of countably many compact sets of finite k-Hausdorff measure*†† *and as such its dimension does not exceed k.*

The proof is based on the following "fishing net" lemma:

LEMMA 1. *Let Z be a countable set linear over the rationals and dense in a Banach space X. Then any closed affine manifold $M^{(k)} \subset X$ of codimension k intersects the union of all k-dimensional manifolds through any $k+1$ points in Z in a dense set in $M^{(k)}$.*

PROOF. Let $M^{(k)} = x_0 + V^{(k)}$, where $V^{(k)}$ is a closed subspace of codimension k, and let $U^{(k)}$ be a k-dimensional subspace spanned by k points in Z such that $V^{(k)} \cap U^{(k)} = \{0\}$. The existence of such a $U^{(k)}$ is a consequence of Z being dense in X. With this choice, $(U^{(k)}, V^{(k)})$ is a couple of complementary closed subspaces in X, and the projection P on $V^{(k)}$ along $U^{(k)}$ is everywhere defined, linear, and continuous. Now, for any $x \in X$,

$$P(x - x_0) \in V^{(k)}, \; P(x - x_0) - (x - x_0) \in U^{(k)},$$

and so

$$Px + (x_0 - Px_0) = P(x - x_0) + x_0 \in (x_0 + V^{(k)}) \cap (x + U^{(k)})$$
$$= M^{(k)} \cap (x + U^{(k)}).$$

Hence, if $z \in Z$,

† That is, contains the intersection of an open ball with a closed affine manifold of codimension k through its center.

†† For the definition and properties of Hausdorff measures we refer the reader to [3, Ch. VII].

$$Pz + (x_0 - Px_0) \in M^{(k)} \cap N^{(k)},$$

where $N^{(k)} = z + U^{(k)}$ is a k-dimensional manifold spanned by $k + 1$ points in Z. Therefore $PZ + (x_0 - Px_0)$ is part of the set in question, and a dense part indeed, because the range of the continuous mapping $x \to Px + (x_0 - Px_0)$ is $M^{(k)}$ and Z is dense in X. Thus the lemma is proved.

PROOF OF THEOREM 2. We may assume that the domain of definition of S is closed, for if not, S could be extended by continuity to its closure. By the lemma above, there is a countable family of k-dimensional manifolds $\{N^{(k)}\}$ such that if $y \in W^{(k)}$ then $S^{-1}y$ intersects their union $\cup N^{(k)}$; in consequence, $W^{(k)} \subset S(\cup N^{(k)})$. Each $N^{(k)}$, being a k-dimensional affine manifold, can be decomposed into a countable number of compact sets with finite k-Hausdorff measure, and as there are only denumerable many $N^{(k)}$'s, the same is true for $\cup N^{(k)}$. Therefore, $W^{(k)}$ is contained in the union of the images of these sets, which, since $D(S)$ is closed and S Lipschitizian, are again compact and of finite k-Hausdorfl measure, and the theorem is proved. The assertion concerning the dimension follows from the fact that the dimension of a set with finite k-Hausdorff measure does not exceed k [3, Ch. VII].

The hypotheses of this theorem can be relaxed a great deal. Indeed, no use has been made of the Banach structure of the space Y, which therefore could be taken as a simple metric space; moreover, a local Lipschitz character is all that the proof demands. It is also apparent that the balls of finite codimension could be replaced by more general objects.

Theorem 2 extends naturally to accretive operators, which by their very definition are directly connected with Lipschitz mappings. Let us recall that a mapping $T: X \to X$ from a Banach space into itself is said to be accretive if for every $\lambda > 0$, $I + \lambda T$ is noncontractive. In Hilbert space, the notions on monotonicity and accretiveness coincide.

THEOREM 3. *Let $T: X \to X$ be an accretive mapping from a separable Banach space into itself. Then the set of x's for which Tx contains a ball of codimension not larger than k is contained in the countable union of compact sets with finite k-Hausdorff measure, and its dimension does not exceed k.*

PROOF. The mapping $S = (I + T)^{-1}$ is nonexpansive. Since the set

$$\{x \in X \mid Tx \text{ contains a ball of codimension} \leqq k\}$$

is the same as the set

$$\{x \in X \mid S^{-1}x = (I + T)x \text{ contains a ball of codimension } \leqq k\},$$

the desired result follows from the previous theorem applied to S.

REMARK. Note that accretiveness has not been used in its full strength, and that only the existence of a λ such that $I + \lambda T$ is locally noncontractive was required.

Now let $X = X^* = H$, and for any closed convex set K in H consider the mappings:

$$x \to V_K x = \{u \mid u \in H, \langle u, x\rangle = \sup_{z \in K} \langle u, z\rangle\}, \ x \in K,$$

$$u \to F_K u = \{x \mid x \in K, \langle u, x\rangle = \sup_{z \in K} \langle u, z\rangle\}, \ u \in H.$$

$V_K x$ is the set of all normals at x and coincides with the dual of the support cone of K at x. We have called it the vertex of K at x [6, Def. 2.4]; its size is a measure of the roughness of K at the point. As to $F_K u$, we have seen that it is the face of K perpendicular to u. Both mappings are monotone and hence accretive, and can be expressed in terms of the projection P_K, which maps any x onto the nearest point in K, as follows:

$$V_K = P_K^{-1} - I, \ F_K = (I - P_K)^{-1} - I,$$

where I is the identity mapping. Theorem 3 applied to them leads to:

COROLLARY 1. *The set of points of a closed convex set K in a separable Hilbert space having a vertex containing a ball of codimension not larger than k is the countable union of compact sets of finite k-Hausdorff measure, and its dimension does not exceed k.*

COROLLARY 2. *The set of normals to a closed convex set K in a separable Hilbert space at faces containing a ball of codimension not larger than k is contained in the countable union of compact sets of finite k-Hausdorff measure, and has dimension not larger than k.*

In the finite dimensional case Corollary 1 reduces to the result of V. L. Klee and R. D. Anderson [1].

REFERENCES

1. R. D. Anderson and V. L. Klee, Jr., *Convex functions and upper semicontinuous collections*, Duke Math. J. **19** (1952), 349–357.

2. N. Dunford and J. Schwartz, *Linear Operators, Part I*, Interscience Publishers, New York, 1956.

3. W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, 1941.

4. S. Mazur, *Über konvexe Menge in linearen normierte Räumen*, Studia Math. **4** (1933), 128–133.

5. R. T. Rockafellar, *Local boundedness of nonlinear monotone operators*, Michigan Math. J. **16** (1969), 397–407.

6. E. H. Zarantonello, *Projections on convex sets in Hilbert space and spectral theory*, in: Contributions to Nonlinear Functional Analysis, E. H. Zarantonello (ed.), Academic Press, New York, 1971, pp. 237–424.

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